Hardy Spaces Generated by an Integrability Condition[1]

S. Fridli

*Department of Numerical Analysis, Eötvös L. University, Budapest, Pázmány P. sétány 1\C,
H-1117 Hungary*
E-mail: fridli@ludens.elte.hu

S. A. Telyakovskiĭ (1964, *Izv. Akad. Nauk. SSR. Ser. Mat.* **28**, 1209–1236) proved an integrability condition for cosine series. No condition superior to that has been given so far. In this paper we identify the atomic structure of the Hardy type space that can be associated with this condition. As a consequence, we conclude that Telyakovskiĭ's condition is equivalent to certain Sidon type inequalities. Then on the basis of this equivalence we show how the atomic technique can be used to extend Telyakovskiĭ's condition to several systems, including Walsh series and integrals, in a uniform way.   © 2001 Academic Press

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1. INTRODUCTION

Let $\mathbf{a} = (a_k)$ be a null sequence of real numbers. The following estimate is due to Telyakovskiĭ [20]

$$\int_0^\pi \left| \sum_{k=0}^\infty a_k \cos kx \right| dx \leqslant C \left(\sum_{k=0}^\infty |\varDelta a_k| + \sum_{n=2}^\infty \left| \sum_{k=1}^{[n/2]} \frac{\varDelta a_{n-k} - \varDelta a_{n+k}}{k} \right| \right), \quad (1)$$

where $\varDelta a_k = a_{k-1} - a_k$ $(k \geqslant 1)$, and $\varDelta a_0 = 0$. (Here and later C will always denote an absolute positive constant not necessarily the same in different occurrences.) We note that the inequality in (1) is strongly related to and is an improvement of an earlier result of Boas [2]. Let us introduce the sequence transform $\mathcal{T}_N$, and call it discrete Telyakovskiĭ transform, as follows

$$(\mathcal{T}_N a)_n = \sum_{k=1}^{[n/2]} \frac{a_{n-k} - a_{n+k}}{k} \qquad (n \geqslant 2).$$

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Set $(\mathcal{T}_N a)_0 = (\mathcal{T}_N a)_1 = 0$. Then (1) can be formulated as

$$\int_0^\pi \left| \sum_{k=0}^\infty a_k \cos kx \right| dx \leqslant C(\|\Delta\mathbf{a}\|_{\ell^1} + \|\mathcal{T}_N(\Delta\mathbf{a})\|_{\ell^1}),$$

where $\Delta\mathbf{a}$ stands for the sequence of differences. It means that if the right side is finite then the pointwise limit of the cosine series represents an integrable function. Several integrability conditions were given for cosine series since (1) had been proved. It turned out that, however, some of them were incomparable with (1) no one was superior to that. For a summary on such conditions and comparison analysis we refer the reader to [1], [3], [15] and [9]. We note that the proof of (1) is quite long, and makes use of special properties of the trigonometric system. That might be why, however, generalizations exist (see e.g. Liflyand [14]) the condition has not yet been extended to other popular orthonormal systems like for example the Walsh system. One of our goals is to overcome this shortage.

We will need the continuous version $\mathcal{T}_{R^+}$, called Telyakovskiĭ transform. It is defined for any locally integrable function $f : R^+ \mapsto R$ by

$$\mathcal{T}_{R^+} f(x) = \int_0^{x/2} \frac{f(x-t) - f(x+t)}{t}\, dt = \int_{x/2}^{3x/2} \frac{f(t)}{x-t}\, dt, \tag{2}$$

where the integral is meant in the Cauchy principal value sense. $\mathcal{T}_{R^+}$ resembles the Hilbert transform $\mathcal{H}$ which is defined as

$$\mathcal{H} f(x) = \int_{-\infty}^\infty \frac{f(t)}{x-t}\, dt = \int_0^\infty \frac{f(x-t) - f(x+t)}{t}\, dt.$$

For technical reasons we omitted the usual $1/\pi$ factor in the definition of $\mathcal{H}$. Even formally it is obvious, that there should be connection between the Telyakovskiĭ and the Hilbert transforms. In order to demonstrate that there is an essential difference between them let us take the characteristic function $\chi_{[0,\delta]}$ ($\delta > 0$). It is known that $\|\mathcal{H}\chi_{[0,\delta]}\|_{L^1(R)} = \infty$. On the other hand,

$$(\mathcal{T}_{R^+}\chi_{[0,\delta]})(x) = \begin{cases} 0 & 0 \leqslant x \leqslant 2\delta/3, \\ \ln(x/2) - \ln|\delta - x| & 2\delta/3 \leqslant x \leqslant 2\delta, \\ 0 & x > 2\delta. \end{cases}$$

Consequently, $\|\mathcal{T}_{R^+}\chi_{[0,\delta]}\|_{L^1(R^+)} = \delta \ln 3$.

2. HARDY SPACES GENERATED BY THE TELYAKOVSKIĬ TRANSFORMS

Let the Hardy spaces $\mathbf{H}_R$, $\mathbf{H}_{R^+}$, and $\mathbf{H}_N$ be defined as

$$\mathbf{H}_R = \{f \in L^1(\boldsymbol{R}) : \mathscr{H}f \in L^1(\boldsymbol{R})\},$$
$$\mathbf{H}_{R^+} = \{f \in L^1(\boldsymbol{R}^+) : \mathscr{T}_{R^+}f \in L^1(\boldsymbol{R}^+)\},$$
$$\mathbf{H}_N = \{\mathbf{a} \in \ell^1 : \mathscr{T}_N\mathbf{a} \in \ell^1\},$$

with the norms

$$\|f\|_{\mathbf{H}_R} = \|f\|_{L^1(\boldsymbol{R})} + \|\mathscr{H}f\|_{L^1(\boldsymbol{R})},$$
$$\|f\|_{\mathbf{H}_{R^+}} = \|f\|_{L^1(\boldsymbol{R}^+)} + \|\mathscr{T}_R + f\|_{L^1(\boldsymbol{R}^+)},$$
$$\|\mathbf{a}\|_{\mathbf{H}_N} = \|a\|_{\ell^1} + \|\mathscr{T}_N a\|_{\ell^1}.$$

In our first result the atomic structure of $\mathbf{H}_{R^+}$ will be characterized. f will be called an $\boldsymbol{R}^+$-atom of

(a) first type if $\mathsf{f} = \delta^{-1}\chi_{[0,\,\delta]}$ with some $\delta > 0$,

(b) second type if there exists a finite interval $I \subset \boldsymbol{R}^+$ such that

(i) $\operatorname{supp} \mathsf{f} \subset I$,   (ii) $\displaystyle\int_I \mathsf{f} = 0$,   (iii) $\|\mathsf{f}\|_{L^\infty(\boldsymbol{R}^+)} \leqslant |I|^{-1}$,

where $|I|$ stands for the length of I. The collection of $\boldsymbol{R}^+$-atoms will be denoted by $\mathscr{A}_{R^+}$. Then our result reads as follows.

THEOREM 2.1.

(i) $f \in \mathbf{H}_{R^+}$ if and only if f can be decomposed as $f = \sum_{k=0}^{\infty} \alpha_k \mathsf{f}_k$, where $\mathsf{f}_k \in \mathscr{A}_{R^+}$, and $\alpha_k \in \boldsymbol{R}$ $(k \in \boldsymbol{N})$ with $(\alpha_k) \in \ell^1$. (The convergence in the decomposition is a. e. and in $L^1(\boldsymbol{R}^+)$ norm.) Moreover

$$\|f\|_{\mathbf{H}_{R^+}} \approx \inf \sum_{k=0}^{\infty} |\alpha_k|,$$

where the infimum is taken over all decompositions of f.

(ii) $\mathbf{H}_{R^+}$ is isomorphic to the subspace of odd functions in $\mathbf{H}_R$.

Remark 2.1. First we note that the equivalence in (ii) was recognized by Liflyand (see e.g. [16], [14]). Actually, it is a consequence of two

integral equalities that can be found in the proof of Theorem 2 in [14]. For the sake of completeness we give a short proof by using these formulas. Part (i), i.e. the atomic characterization of $\mathbf{H}_{R^+}$ is based on the isomorphism in (ii). In connection with it we want to call the attention to a similar situation. Namely, the real nonperiodic Hardy space on $[0,1)$ is isomorphic to the subspace of even functions of the real periodic Hardy space on $[-\pi, \pi]$ (see e.g. [13, Chapter 5]). The importance of the equivalence in (i) is that it allows to use atomic technique in proofs and then express the results in a closed form by the transform.

Since $\mathbf{H}_N$ is defined by $\mathcal{T}_N$, the discrete analogue of $\mathcal{T}_{R^+}$, it can be considered as the discrete version of $\mathbf{H}_{R^+}$. There are, however, at least two other natural ways to introduce such a space.

Namely, let $\mathcal{P}\mathbf{a}$ denote the step function associated to the real sequence $\mathbf{a}$ by

$$(\mathcal{P}\mathbf{a})(x) = a_{[x]} \qquad (x \in \mathbf{R}^+),$$

where $[x]$ stands for the integer part of x. Then following the scheme $\mathbf{a} \in \ell^p$ $(1 \leqslant p \leqslant \infty)$ if and only if $\mathcal{P}\mathbf{a} \in L^p(\mathbf{R}^+)$, and $\|\mathbf{a}\|_{\ell^p} = \|\mathcal{P}\mathbf{a}\|_{L^p(\mathbf{R}^+)}$ we can introduce a discrete version of $\mathbf{H}_{R^+}$.

Another way to define the discrete space is based on the concept of atomic decomposition. Let the real sequence $\mathbf{a}$ be called an N-atom if $\mathcal{P}\mathbf{a}$ is an $\mathbf{R}^+$-atom. The collection of N-atoms is denoted by $\mathcal{A}_N$. Then a discrete analogue of $\mathbf{H}_{R^+}$ can be introduced by N-atoms.

The following theorem shows that no matter which way we choose we get to the same space.

THEOREM 2.2. *Let* $\mathbf{a}$ *be a real sequence. Then the following statements are equivalent.*

   (i) $\mathbf{a} \in \mathbf{H}_N$,

   (ii) $\mathcal{P}\mathbf{a} \in \mathbf{H}_{R^+}$,

   (iii) $\mathbf{a}$ *can be decomposed as* $\mathbf{a} = \sum_{k=0}^{\infty} \alpha_k \mathbf{a}^{(k)}$, *where* $\mathbf{a}^{(k)} \in \mathcal{A}_N$, *and* $\alpha_k \in \mathbf{R}$ $(k \in N)$ *with* $(\alpha_k) \in \ell^1$. *(The convergence in the decomposition is in* ℓ^1 *norm.)*

Moreover

$$\|\mathcal{P}\mathbf{a}\|_{\mathbf{H}_{R^+}} \approx \|\mathbf{a}\|_{\mathbf{H}_N} \approx \inf \sum_{k=0}^{\infty} |\alpha_k|,$$

where the infimum is taken over all decompositions of $\mathbf{a}$.

Remark 2.2. We note that a modification of the reasoning used by Liflyand in the proof of Theorem 5 in [14] would actually prove the equivalence of (i) and (ii). Here, we will give a proof by only using elementary estimates.

3. INTEGRABILITY CONDITIONS

In this section first we will apply *Theorem 2.1*, especially the atomic decomposition of $\mathbf{H}_{R^+}$, to show that (1) is equivalent to a Sidon type inequality. As a consequence, we obtain that Telyakovskiĭ's condition can be extended to several orthonormal systems in a uniform way. Then we give a continuous version of our result.

Let $\Phi = (\phi_k)$ be an orthonormal system defined on $[0, 1]$ whose terms are in $L^\infty[0, 1]$. Set $D_k^\Phi = \sum_{j=0}^{k-1} \phi_j$, $D_0^\Phi \equiv 0$ ($k \in N$). Our theorem reads as follows.

THEOREM 3.1. *Suppose that there exists a constant C_Φ such that*

$$\int_0^1 \left| \sum_{k=0}^\infty \mathsf{a}_k D_k^\Phi(x) \right| dx \leqslant C_\Phi \qquad (\mathsf{a} \in \mathscr{A}_N). \tag{3}$$

Then for any null-sequence $\mathbf{a}$ *with* $\varDelta\mathbf{a} \in \mathbf{H}_N$ *the Φ-series $\sum_{k=0}^\infty a_k\phi_k$ is the Φ-Fourier-series of an $f \in L^1[0, 1]$, and $\|f\|_{L^1[0, 1]} \leqslant C_\Phi \|\varDelta\mathbf{a}\|_{\mathbf{H}_N}$.*

If in addition

$$\sup_{k \in N} |D_k^\Phi(x)| < \infty \qquad (a.e.\ x \in [0, 1]) \tag{4}$$

then f is the pointwise limit of $\sum_{k=0}^\infty a_k\phi_k$.

Remark 3.1. First we remark that inequality (3) is a so called Sidon type inequality. The study of them has a long history. We refer the reader to [9] for details. Theorem 3.1 shows that a Sidon type inequality implies Telyakovskiĭ's condition for the system Φ. If (4) holds for Φ then also the converse is true as one can easily show it by summation by parts.

Now we take two examples, the *cosine* and the *Walsh systems*. Inequality (3), in a different form, was proved for the trigonometric Dirichlet kernels by *Schipp* in [17]. Namely, first he associated a step function on $[0, 1]$ with every finite sequence. Then he proved the corresponding inequality by using the norm of the real non-periodic Hardy space on $[0, 1]$. If we translate his result into our terminology then it turns out that it is exactly (3).

We note that the trigonometric Dirichlet kernels satisfy condition (4). Therefore, we can conclude by *Theorem 3.1* that the Telyakovskiĭ's and Schipp's results are equivalent.

Also, Schipp proved (3) for the Walsh–Dirichlet kernels with some restrictions in [17]. Namely, he employed the dyadic Hardy space on [0, 1] in his proof and showed (3) for dyadic atoms. The author showed in [11] that Schipp's result can be extended to any *N*-atom. We note that, as we proved in [7] and [8], (3) is best possible in a sense for the *Walsh* and the *cosine* systems. Since also the Walsh–Dirichlet kernels satisfy condition (4) we have the following corollary.

COROLLARY 3.1. *Let $\Phi = (\phi_k)$ stand for the cosine or the* Walsh *system. Suppose that* $\mathbf{a}$ *is a null sequence such that* $\Delta\mathbf{a} \in \mathbf{H}_N$. *Then* $\sum_{k=0}^{\infty} a_k \phi_k$ *is Φ-Fourier series and*

$$\int_0^1 \left| \sum_{k=0}^{\infty} a_k \phi_k(x) \right| dx \leqslant C \left(\sum_{n=0}^{\infty} |\Delta a_n| + \sum_{n=2}^{\infty} \left| \sum_{k=1}^{[n/2]} \frac{\Delta a_{n-k} - \Delta a_{n+k}}{k} \right| \right).$$

Since a Sidon type inequality holds for several other systems, including *UDMD, Ciesielski,* certain polynomial systems (see [17], [18]) etc., an inequality similar to that in *Corollary 3.1* can be formalized for them as well.

The following theorem can be considered as the continuous version of *Corollary 3.1*. For the sake of compactness we confine ourselves to two basic models, the *cosine* and the *Walsh* integrals. Let Ψ denote either the *cosine* or the *Walsh* system on $\mathbf{R}^+$. The Ψ transform $\hat{g}^{\Psi}$ of a $g \in L^1(\mathbf{R}^+)$ is defined as $\hat{g}^{\Psi}(t) = \int_0^{\infty} g(x) \psi_t(x) \, dx$ $(t \in \mathbf{R}^+)$. Then the following integrability theorem holds for Ψ.

THEOREM 3.2. *Let $f : \mathbf{R}^+ \mapsto \mathbf{R}$ be a locally absolutely continuous function with $f' \in \mathbf{H}_{\mathbf{R}^+}$, and* $\lim_{t \to \infty} f(t) = 0$. *Then*

$$\|g\|_{L^1(\mathbf{R}^+)} \leqslant C \|f'\|_{\mathbf{H}_{\mathbf{R}^+}}, \qquad \text{and} \qquad \hat{g}^{\Psi} = f,$$

where $g(x) = \lim_{u \to \infty} \int_0^u f(t) \psi_x(t) \, dt$.

We note that the *cosine* case of *Theorem 3.2* was proved by Liflyand [14] by adapting Telyakovskiĭ's method for *cosine* integrals. Also, it was partially claimed and proved by Giang and Móricz in [12]. We are aware that in some cases, for instance in the proof of the inversion formula in *Theorem 3.2*, one can refer to earlier results or can take shortcuts by using special properties of the trigonometric system. Our aim is, however, to

demonstrate how the atomic technique makes it possible to provide a uniform treatment for the two systems. In order to do that we will need the following Sidon type inequality in which $D_t^\Psi(x) = \int_0^t \psi_u(x)\,du$ $(x, t \in \mathbf{R}^+)$ denotes the Ψ-Dirichlet kernel.

THEOREM 3.3. *Let Ψ denote either the* Walsh *or the cosine system on $\mathbf{R}^+$. Then*

$$\int_{\mathbf{R}^+} \left| \int_{\mathbf{R}^+} h(t)\, D_t^\Psi(x)\, dt \right| dx \leqslant C\, \|h\|_{\mathbf{H}_{\mathbf{R}^+}} \qquad (h \in \mathbf{H}_{\mathbf{R}^+}). \tag{5}$$

4. PROOFS

Proof of Theorem 2.1. First we show (ii). Let $f \in L^1(\mathbf{R}^+)$ and let f_O denote the odd extension of $f \in L^1(\mathbf{R}^+)$, i.e.

$$f_O(x) = \begin{cases} f(x) & x \geqslant 0 \\ -f(-x) & x < 0. \end{cases}$$

Since also $\mathscr{H}f_O$ is odd it is enough to prove that $\int_0^\infty |\mathscr{H}f_O - \mathscr{T}_{\mathbf{R}^+} f| \leqslant C \int_0^\infty |f|$. By definition

$$\mathscr{H}f_O(x) - \mathscr{T}_{\mathbf{R}^+} f(x) = 2(I_1 f(x) + I_2 f(x)) - I_3 f(x) \qquad (x > 0),$$

where

$$I_1 f(x) = \int_0^{x/2} f(t)\, \frac{t}{x^2 - t^2}\, dt,$$

$$I_2 f(x) = \int_{3x/2}^\infty f(t)\, \frac{t}{x^2 - t^2}\, dt,$$

$$I_3 f(x) = \int_{x/2}^{3x/2} \frac{f(t)}{x + t}\, dt.$$

A direct calculation shows (see [14]) that

$$\|I_1 f\|_{L^1(\mathbf{R}^+)} \leqslant \int_0^\infty \int_0^{x/2} |f(t)|\, \frac{t}{x^2 - t^2}\, dt\, dx = \ln\sqrt{3}\, \|f\|_{L^1(\mathbf{R}^+)},$$

$$\|I_2 f\|_{L^1(\mathbf{R}^+)} \leqslant \int_0^\infty \int_{3x/2}^\infty |f(t)|\, \frac{t}{x^2 - t^2}\, dt\, dx = \ln\sqrt{5}\, \|f\|_{L^1(\mathbf{R}^+)}.$$

On the other hand $\|I_3 f\|_{L^1(R^+)} \leqslant \ln(5/3) \|f\|_{L^1(R^+)}$. Therefore, $f \in \mathbf{H}_{R^+}$ if and only if $f_O \in \mathbf{H}_R$, and $\|f\|_{\mathbf{H}_{R^+}} \approx \|f_O\|_{\mathbf{H}_R}$.

Now we will show (i), i.e. that $\mathbf{H}_{R^+}$ has an atomic structure. Our proof will be based on the isomorphism in (ii) and on the atomic structure of $\mathbf{H}_R$ (see [5] or [6]). A function $\mathsf{g} \in L^\infty(R)$ is called an R-atom, in notation $\mathsf{g} \in \mathscr{A}_R$, if there exists a finite interval $I \subset R$ such that

$$\text{(i)} \quad \operatorname{supp} \mathsf{g} \subset I, \qquad \text{(ii)} \quad \int_I \mathsf{g} = 0, \qquad \text{(iii)} \quad \|\mathsf{g}\|_{L^\infty(R)} \leqslant |I|^{-1}.$$

Then an atomic decomposition similarly to what is described in (i) of Theorem 2.1 for $\mathbf{H}_{R^+}$ holds for every $\mathsf{g} \in \mathbf{H}_R$ with $\mathscr{A}_{R^+}$ replaced by $\mathscr{A}_R$.

Since $\|f_O\|_{\mathbf{H}_R} \approx \|f\|_{\mathbf{H}_{R^+}}$ we only need to show that to any decomposition $\sum_{k=0}^\infty \alpha_k \mathsf{g}_k$ ($\mathsf{g}_k \in \mathscr{A}_R, k \in N$) of f_O there corresponds a decomposition $\sum_{k=0}^\infty \beta_k \mathsf{f}_k$ ($\mathsf{f}_k \in \mathscr{A}_{R^+}, k \in N$) of f for which $\sum_{k=0}^\infty |\beta_k| \leqslant C \sum_{k=0}^\infty |\alpha_k|$, and conversely.

Let $\sum_{k=0}^\infty \alpha_k \mathsf{g}_k$ be an atomic decomposition of f_O, and set $f_k(x) = (\mathsf{g}_k(x) - \mathsf{g}_k(-x))$ ($x \in R, k \in N$). Then $f_O = \sum_{k=0}^\infty \frac{1}{2} \alpha_k f_k$ since f_O is odd. Moreover, $f = \sum_{k=0}^\infty \frac{1}{2} \alpha_k f_k^+$, where $f_k^+ = f_k \chi_{[0,\infty]}$. Let $I_k \subset R$ be a finite interval such that $\operatorname{supp} \mathsf{g}_k \subset I, \|\mathsf{g}_k\|_{L^\infty(R)} \leqslant |I_k|^{-1}$, and define $\mathscr{N}_1 \subset N$ as the collection of k's for which $0 \notin I_k$. Then $f_k^+ \in \mathscr{A}_{R^+}$ for any $k \in \mathscr{N}_1$.

If, on the other hand, $k \in \mathscr{N}_2 = N \setminus \mathscr{N}_1$ then there exists a minimal $\delta > 0$ for which $\operatorname{supp} g_k \subset [-\delta, \delta]$. Thus $\|f_k^+\|_{L^\infty(R^+)} \leqslant 2 \|\mathsf{g}_k\|_{L^\infty(R)} \leqslant 2\delta^{-1}$. Set

$$\mathsf{f}_k = \frac{1}{4}\left(f_k^+ - \left(\int_0^\delta f_k^+ \right) \frac{1}{\delta} \chi_{[0,\delta]} \right), \quad \mathsf{h}_k = \frac{1}{\delta} \chi_{[0,\delta]} \qquad (k \in \mathscr{N}_2).$$

Then $f_k^+ = 4\mathsf{f}_k + (\int_0^\delta f_k^+) \mathsf{h}_k$, where h_k is a first type and f_k is a second type R^+-atom. Consequently,

$$f = \sum_{k \in \mathscr{N}_1} \frac{1}{2} \alpha_k f_k^+ + \sum_{k \in \mathscr{N}_2} 2\alpha_k \mathsf{f}_k + \sum_{k \in \mathscr{N}_2} \left(\frac{1}{2} \alpha_k \int_0^\delta f_k^+ \right) \mathsf{h}_k$$

is a proper atomic decomposition of f.

For the other direction let $f = \sum_{k=0}^\infty \alpha_k \mathsf{f}_k$ be an atomic decomposition of an $f \in \mathbf{H}_{R^+}$. Then $f_O = \sum_{k=0}^\infty \alpha_k (\mathsf{f}_k)_O$. The proof can be finished by noting that if f_k is a first type R^+-atom then $1/2(\mathsf{f}_k)_O$ is an R-atom while if f_k is a second type R^+-atom then $(\mathsf{f}_k)_O$ is a sum of two R- atoms. ∎

Proof of Theorem 2.2.   Let us start with the equivalence of (i) and (ii). By definition $\|\mathbf{a}\|_{\ell^1} = \|\mathscr{P}\mathbf{a}\|_{L^1(R^+)}$ and $\|\mathscr{T}_N \mathbf{a}\|_{\ell^1} = \|\mathscr{P}(\mathscr{T}_N \mathbf{a})\|_{L^1(R^+)}$.

Therefore we only need to prove that

$$\|\mathcal{T}_{R^+}(\mathcal{P}\mathbf{a}) - \mathcal{P}(\mathcal{T}_N\mathbf{a})\|_{L^1(R^+)} \leqslant C \|\mathbf{a}\|_{\ell^1}. \tag{6}$$

Write $x = n + \delta$ $(n \in N, 0 < \delta < 1)$ and subdivide the interval of integration in $\mathcal{T}_{R^+}(\mathcal{P}\mathbf{a})$ according to the terms of $\mathbf{a}$. Then after grouping the corresponding terms we will have two main sums and a remainder. Namely,

$$\mathcal{T}_{R^+}(\mathcal{P}\mathbf{a})(x) - \mathcal{P}(\mathcal{T}_N\mathbf{a})(x) = A_1(x) + A_{-1}(x) + R(x),$$

where

$$A_i(x) = \sum_{k=1}^{[n/2]} a_{n+ik} \left(\int_{k+i(1-\delta)}^{k-i\delta} \frac{1}{t}\, dt + \frac{i}{k} \right) \qquad (i = \pm 1).$$

Elementary calculation shows that

$$|A_{\pm 1}(x)| \leqslant \sum_{k=2}^{[n/2]} \frac{2}{k^2} |a_{n\pm k}| + |a_{n\pm 1}|\, \omega(\delta)$$

with $\omega(\delta) = |\ln \delta| + |\ln(1-\delta)| + 2$. For the remainder $R(x)$ we have

$$|R(x)| \leqslant \left(|a_n| + \sum_{j=0}^{1} (|a_{n+[n/2]+j}| + |a_{n-[n/2]-j}|) \right) \omega(\delta).$$

Note that $\int_0^1 \omega(\delta)\, d\delta = 4$. Therefore, by integrating $|A_{\pm 1}|$ and R first on $[n, n+1]$ with respect to δ then summing on n we obtain (6).

In this section let the $\mathbf{H}_{R^+}$, and $\mathbf{H}_N$ norms considered to be defined via the atomic decompositions. Suppose that $\mathbf{a}$ is a real sequence that can be decomposed as $\mathbf{a} = \sum_{k=0}^{\infty} \alpha_k \mathbf{a}^{(k)}$ $(\mathbf{a}^{(k)} \in \mathscr{A}_N, (\alpha_k) \in \ell^1)$. Then $\mathcal{P}\mathbf{a} \in \mathbf{H}_{R^+}$, and $\|\mathcal{P}\mathbf{a}\|_{\mathbf{H}_{R^+}} \leqslant \inf \sum_{k=0}^{\infty} |\alpha_k|$ are straightforward by the definition of N-atoms. Therefore, (iii) implies (ii).

For the proof of the other direction define $\mathscr{E}f$ as follows

$$\mathscr{E}f(x) = \int_{[x]}^{[x]+1} f \qquad (f \in L^1(R^+), x \in R^+).$$

Let $\mathsf{f} \in \mathscr{A}_{R^+}$. We will consider $\mathscr{E}f$. If f is a second type atom then let $I \in R^+$ be such a finite interval for which supp $\mathsf{f} \subset I$, and $\|\mathsf{f}\|_{L^\infty(R^+)} \leqslant |I|^{-1}$. Then there exists $[n, n+k] \supset I$ $(n, k \in N)$ with minimal length. Thus $\int_0^\infty \mathscr{E}\mathsf{f} = \int_I \mathsf{f} = 0$, and supp $\mathscr{E}\mathsf{f} \subset [n, n+k]$. Note that if $k = 1$ then $\mathscr{E}\mathsf{f} \equiv 0$. Moreover, $\|\mathscr{E}\mathsf{f}\|_{L^\infty(R^+)} \leqslant \min\{1, \|\mathsf{f}\|_{L^\infty(R^+)}\}$, and $k \leqslant |I| + 2$. Hence we have $1/3\, \mathscr{E}\mathsf{f} \in \mathscr{A}_{R^+}$.

Let f be a first type atom, i.e. $f = \delta^{-1}\chi_{[0,\delta]}$ with some $\delta > 0$. If $\delta \leqslant 1$ then $\mathscr{E}f = \chi_{[0,1]} \in \mathscr{A}_{R^+}$. For $\delta > 1$ let us take the decomposition $f = f_1 + f_2$, where

$$f_1 = ([\delta]+1)^{-1}\chi_{[0,[\delta]+1]},$$
$$f_2 = (\delta^{-1} - ([\delta]+1)^{-1})\chi_{[0,\delta]} - ([\delta]+1)^{-1}\chi_{[\delta,[\delta]+1]}.$$

Clearly, f_1 is a first type, and f_2 is a second type atom. Hence $\mathscr{E}f = \mathscr{E}f_1 + 3\mathscr{E}(1/3\,f_2)$, where $\mathscr{E}f_1$, $\mathscr{E}(1/3\,f_2) \in \mathscr{A}_{R^+}$.

By definition, if $\mathscr{E}g \in \mathscr{A}_{R^+}$ then there exists $b \in \mathscr{A}_N$ such that $\mathscr{P}b = \mathscr{E}g$. Consequently, in view of $\mathscr{P}a = \mathscr{E}(\mathscr{P}a)$, any atomic decomposition of $\mathscr{P}a$ in H_{R^+} can naturally be associated with a decomposition of a in H_N, and $\|a\|_{H_N} \leqslant 4\|\mathscr{P}a\|_{H_{R^+}}$.   ∎

Proof of Theorem 3.1.   Let a be a null-sequence such that $\varDelta a \in H_N$. Moreover, let $\varDelta a = \sum_{k=0}^{\infty} \alpha_k a^{(k)}$ be an atomic decomposition of $\varDelta a$. Since $\lim_{n\to\infty}\|\varDelta a - \sum_{k=0}^{n}\alpha_k a^{(k)}\|_{\ell^1} = 0$ we have that

$$\varDelta a_j = \sum_{k=0}^{\infty} \alpha_k a_j^{(k)}. \tag{7}$$

We note that the convergence is absolute and uniform in j. Indeed, $a^{(k)} \in \mathscr{A}_N$ implies $\mathscr{P}a^{(k)} \in \mathscr{A}_{R^+}$. Then by definition there exists a finite interval $I \subset R^+$ such that $\operatorname{supp}\mathscr{P}a^{(k)} \subset I$, and $\|\mathscr{P}a^{(k)}\|_{L^{\infty}(R^+)} \leqslant |I|^{-1}$. Obviously, $|I|^{-1} \leqslant 1$. Hence $|a_j^{(k)}| \leqslant \|\mathscr{P}a^{(k)}\|_{L^{\infty}(R^+)} \leqslant 1$. Consequently, $\sum_{k=n}^{\infty}|\alpha_k a_j^{(k)}| \leqslant \sum_{k=n}^{\infty}|\alpha_k| \to 0$ as $n \to \infty$.

Let us consider the series

$$\sum_{k=0}^{\infty} \alpha_k \sum_{j=0}^{\infty} a_j^{(k)} D_j^{\Phi}. \tag{8}$$

(3) implies that this series converges to an f in $L^1[0,1]$ with $\|f\|_{L^1[0,1]} \leqslant C_{\Phi}\|\varDelta a\|_{H_N}$. (Here the H_N norm is defined via the atomic decompositions instead of the discrete Telyakovskiĭ transform.) By definition

$$\hat{f}(\ell) = \int_0^1 f\phi_\ell = \int_0^1 \sum_{k=0}^{\infty} \alpha_k \sum_{j=0}^{\infty} a_j^{(k)} D_j^{\Phi}\phi_\ell. \tag{9}$$

The inner sum has finite many terms since $a^{(k)} \in \mathscr{A}_N$. Moreover, we have by (3) that

$$\int_0^1 \left|\sum_{j=0}^{\infty} a_j^{(k)} D_j^{\Phi}\phi_\ell\right| \leqslant C_{\Phi}\|\phi_\ell\|_{L^{\infty}[0,1]}.$$

Therefore it follows from $(\alpha_k) \in \ell^1$ that we may interchange the integration and the summations to obtain

$$\hat{f}(\ell) = \sum_{k=0}^{\infty} \sum_{j=\ell+1}^{\infty} \alpha_k \mathsf{a}_j^{(k)}.$$

This double series is absolutely convergent. Indeed, by the definition of N-atoms we have $\sum_{j=0}^{\infty} |\mathsf{a}_j^{(k)}| \leqslant 1$. Hence

$$\sum_{k=0}^{\infty} \sum_{j=\ell+1}^{\infty} |\alpha_k| \, |\mathsf{a}_j^{(k)}| \leqslant \sum_{k=0}^{\infty} |\alpha_k| < \infty. \tag{10}$$

Then it follows from (7) that

$$\hat{f}(\ell) = \sum_{j=\ell+1}^{\infty} \sum_{k=0}^{\infty} \alpha_k \mathsf{a}_j^{(k)} = \sum_{j=\ell+1}^{\infty} \varDelta a_j = a_\ell.$$

The first part of *Theorem 3.1* is proved.

Set $C(x) = \sup_{k \in N} |D_k^{\Phi}(x)|$ $(x \in [0, 1])$ and suppose that $C(x) < \infty$ for a.e. $x \in [0, 1]$. By summation by parts we obtain

$$\sum_{k=0}^{n} a_k \phi_k(x) = \sum_{k=0}^{n} \varDelta a_k D_k^{\Phi}(x) + a_n D_{n+1}^{\Phi}(x) \qquad (x \in [0, 1], n \in N).$$

Note that $\mathbf{a}$ is a null-sequence. Moreover, $\varDelta \mathbf{a} \in \mathbf{H}_N$ implies that $\mathbf{a}$ is of bounded variation. Consequently, $\sum_{k=0}^{\infty} a_k \phi_k(x)$ exists for any $x \in [0, 1]$ with $C(x) < \infty$, and $\sum_{k=0}^{\infty} a_k \phi_k(x) = \sum_{k=0}^{\infty} \varDelta a_k D_k^{\Phi}(x)$.

Recall that f was defined by the $L^1[0, 1]$ limit of the series in (8). Now we show that the series in (8) converges to $\sum_{k=0}^{\infty} a_k \phi_k(x)$ for a.e. $x \in [0, 1]$. Since the pointwise and the $L^1[0, 1]$ limits coincide if both exist we conclude that $f = \sum_{k=0}^{\infty} a_k \phi_k$ almost everywhere.

Indeed, by (10) we have

$$\sum_{j=0}^{\infty} |\alpha_j| \sum_{k=0}^{\infty} |\mathsf{a}_k^{(j)}| \, |D_k^{\Phi}(x)| \leqslant C(x) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\alpha_j| \, |\mathsf{a}_k^{(j)}| < \infty.$$

Consequently,

$$\sum_{j=0}^{\infty} \alpha_j \sum_{k=0}^{\infty} \mathsf{a}_k^{(j)} D_k^{\Phi}(x) = \sum_{k=0}^{\infty} D_k^{\Phi}(x) \sum_{j=0}^{\infty} \alpha_j \mathsf{a}_k^{(j)} = \sum_{k=0}^{\infty} \varDelta a_k D_k^{\Phi}(x)$$

$$= \sum_{k=0}^{\infty} a_k \phi_k(x). \quad \blacksquare$$

For the proof of *Theorem 3.3* we need two lemmas.

LEMMA 4.1.   *Let* $D_t^{T^+}(x) = \int_0^t e^{2\pi i s x}\, ds = (e^{2\pi i t x} - 1)/(2\pi i x)$ $(0 < x, t < \infty)$. *Then*

$$\int_{R^+} \left| \int_{R^+} \mathsf{h}(t)\, D_t^{T^+}(x)\, dt \right| dx \leqslant C$$

holds for any $\mathsf{h} \in \mathscr{A}_{R^+}$ *of second type.*

Proof.   We will adapt Schipp's method that he used for the $e^{2\pi i n x}$ $(n \in N)$ system in [17]. Let h be an R^+-atom of second type, with supp $\mathsf{h} \subset [u, v]$, and $\|\mathsf{h}\|_{L^\infty(R^+)} \leqslant 1/(v-u)$. We may suppose that $u = 0$. Indeed, $\mathsf{h}^{\#}(t) = \mathsf{h}(u+t)$ $(t \in R^+)$ is an R^+-atom of second type, and it is easy to see that $\int_u^v \mathsf{h} = 0$ implies

$$\int_u^v \mathsf{h}(t)\, D_t^{T^+}(x)\, dt = e^{2\pi i u} \int_0^{v-u} \mathsf{h}^{\#}(t)\, D_t^{T^+}(x)\, dt \qquad (x, t \in R^+).$$

Let now h denote an R^+-atom with supp $\mathsf{h} \subset [0, v]$, $\|\mathsf{h}\|_{L^\infty(R^+)} \leqslant 1/v$, $\int_0^v \mathsf{h} = 0$. Thus

$$\begin{aligned}
\int_{R^+} \left| \int_{R^+} \mathsf{h}(t)\, D_t^{T^+}(x)\, dt \right| dx &= \int_0^{1/v} \left| \int_0^v \mathsf{h}(t)\, D_t^{T^+}(x)\, dt \right| dx \\
&\quad + \int_{1/v}^\infty \left| \int_0^v \mathsf{h}(t)\, D_t^{T^+}(x)\, dt \right| dx \\
&= I_1 + I_2.
\end{aligned}$$

Since $|D_t^{T^+}(x)| \leqslant v$ $(0 \leqslant t \leqslant v)$ we have

$$I_1 \leqslant \int_0^{1/v} \int_0^v v\, |\mathsf{h}(t)|\, dt\, dx = \|\mathsf{h}\|_{L^1(R^+)} \leqslant 1.$$

Set $I_2^\delta = \int_0^\delta \chi_{[1/v, \infty)}(x)\, |\int_0^v h(t)\, D_t^{T^+}(x)\, dt|\, dx$ $(\delta > 0)$. If

$$g(x) = \frac{\chi_{[1/v, \infty)}(x)}{2\pi x} \operatorname{sgn}\left(\int_0^v \mathsf{h}(t)\, e^{2\pi i t x}\, dt \right),$$

where $\operatorname{sgn}(z) = \bar{z}/|z|$ $(z \in \mathbf{C}, z \neq 0, \bar{z}$ is the complex conjugate of z), then I_2^δ can be written as

$$I_2^\delta = \int_0^\delta g(x) \int_0^v h(t)\, e^{2\pi i t x}\, dt\, dx.$$

After changing the order of the integrations we have

$$I_2^\delta = \int_0^v h(t)(\widehat{\chi_{[0,\delta]} g})(t) \, dt.$$

Let us apply first Cauchy–Schwarz inequality and then Parseval equality to obtain

$$I_2^\delta \leqslant \|h\|_{L^2(R^+)} \|\chi_{[0,\delta]} g\|_{L^2(R^+)} \leqslant \frac{1}{2\pi} \frac{1}{v^{1/2}} \left(\int_{1/v}^\infty \left(\frac{1}{x} \right)^2 dx \right)^{1/2} = \frac{1}{2\pi},$$

which is independent of δ.   ∎

Before stating the next lemma we introduce the concept of generalized Walsh functions. To this order let us take the binary expansion of $x \in [0, \infty)$ defined as $x = \sum_{j=-\infty}^\infty x_j 2^{-j-1}$, where $x_j = 0$ or 1. In case when there are two expansions of this form, i.e. in case of dyadic rationals, we take the one that terminates in 0's. The functions

$$w_y(x) = (-1)^{\sum_{j=-\infty}^\infty x_j y_{-j-1}} \qquad (0 \leqslant x, y < \infty) \tag{11}$$

are called generalized Walsh functions. For any two nonnegative numbers x, y their dyadic sum is defined by

$$x \dotplus y = \sum_{j=-\infty}^\infty |x_j - y_j| \, 2^{-j-1}.$$

Then it is clear by (11) that

$$w_t(x) \, w_t(y) = w_t(x \dotplus y) \qquad (0 \leqslant x, y < \infty) \tag{12}$$

provided $x \dotplus y$ is dyadic irrational. Recall that the generalized Dirichlet kernels are defined as $D_t^W(x) = \int_0^t w_u(x) \, du \ (0 \leqslant t, x < \infty)$. Concerning the basic properties of the generalized Walsh functions and Walsh–Dirichlet kernels we refer the reader to [19].

LEMMA 4.2.   *Set*

$$h_{k,n} = 2^{-n-1}(\chi_{[k2^n,(k+1)2^n]} - \chi_{[(k+1)2^n,(k+2)2^n]}) \qquad (k \in N, n \in Z). \tag{13}$$

Then

$$\int_{R^+} \left| \int_{R^+} h_{k,n}(t) \, D_t^W(x) \, dt \right| dx \leqslant C \qquad (k \in N, n \in Z). \tag{14}$$

Proof. By definition $w_{k2^n+t} = w_{k2^n}w_t$ $(k \in N, n \in \mathbf{Z}, 0 \leqslant t < 2^n)$. Therefore, $D_{k2^n+t}^W = D_{k2^n}^W + w_{k2^n}D_t^W$, and $D_{(k+1)2^n+t}^W = D_{k2^n}^W + w_{k2^n}D_{2^n}^W + w_{(k+1)2^n}D_t^W$. Then

$$\int_{R^+} \mathsf{h}_{k,n}(t)\, D_t^W(x)\, dt = 2^{-n-1} \left(\int_{k2^n}^{(k+1)2^n} D_t^W(x)\, dt - \int_{(k+1)2^n}^{(k+2)2^n} D_t^W(x)\, dt \right)$$

$$= 2^{-n-1} \left(w_{k2^n}(x) \int_0^{2^n} D_t^W(x)\, dt - 2^n w_{k2^n}(x)\, D_{2^n}^W(x) \right.$$

$$\left. - w_{(k+1)2^n}(x) \int_0^{2^n} D_t^W(x)\, dt \right).$$

Hence

$$\left| \int_0^\infty \mathsf{h}_{k,n}(t)\, D_t^W(x)\, dt \right| \leqslant \frac{1}{2^n} \left| \int_0^{2^n} D_t^W(x)\, dt \right| + \frac{1}{2} D_{2^n}^W(x).$$

The first term is the generalized Walsh–Fejér kernel $K_{2^n}^W$. In [10] the author proved that for any $h \in L_{\text{loc}}^\infty(\mathbf{R}^+)$ the following inequality holds

$$\int_{R^+} \frac{1}{2^n} \left| \int_0^{2^n} h(t)\, D_t^W(x)\, dt \right| dx \leqslant \|\chi_{[0,\,2^n)}h\|_{L^\infty(\mathbf{R}^+)} \qquad (n \in \mathbf{Z}).$$

In particular, choosing $h = \chi_{[0,\,u]}$ and $n \in \mathbf{Z}$ so that $2^{n-1} < u \leqslant 2^n$ we obtain

$$\sup_{u>0} \|K_u^W\|_{L^1(\mathbf{R}^+)} < \infty. \tag{15}$$

Moreover (see e.g. [19]), $\|D_{2^n}^W\|_{L^1(\mathbf{R}^+)} = 1$. Consequently, (14) holds for any $k \in N, n \in \mathbf{Z}$. ∎

Proof of Theorem 3.3. The left side of (5) is subadditive. Therefore, it is enough to show that (5) holds for any $\mathbf{R}^+$-atom.

Let us start with the *cosine* case. Since $D_t^T(x) = \int_0^t \cos ux\, du$ is the real part of $D_t^{T^+}(x)$ $(t, x \in \mathbf{R}^+)$ we have that (5) follows for second type atoms from Lemma 4.1. If $\mathsf{h} \in \mathscr{A}_{\mathbf{R}^+}$ is of first type then $\mathsf{h} = \frac{1}{u}\chi_{[0,\,u]}$ with some $u > 0$, and $K_u^T(x) = \frac{1}{u}\int_0^u D_t^T(x)\, dt = \int_{R^+} \mathsf{h}(t)\, D_t^T(x)\, dt$ is the generalized trigonometric Fejér kernel. Thus for such atoms (5) is equivalent to $\sup_{u>0} \|K_u^T\|_{L^1(\mathbf{R}^+)} < \infty$ which is a known property (see e.g. [4]) of the trigonometric Fejér kernels.

In case of the *Walsh* system see (15) for the first type atoms. Taking second type atoms we note that it was shown by the author in [10] that (5) holds for dyadic second type atoms. Namely, for atoms h for which supp $\mathsf{h} \subset [k2^n, (k+1) 2^n)$, and $\|\mathsf{h}\|_{L^\infty(R^+)} \leqslant 2^{-n}$ with proper $k \in N, n \in Z$. Moreover, by *Lemma 4.2* we have that (5) holds for atoms defined in (13). Let now h be an arbitrary second type atom. Since the left side of (5) is subadditive it is enough to show that h can be decomposed as a sum of dyadic atoms and atoms of type in (13). Let $I \subset R^+$ be a finite interval for which supp $\mathsf{h} \subset I$, $\|\mathsf{h}\|_{L^\infty(R^+)} \leqslant |I|^{-1}$, and $\int_I \mathsf{h} = 0$. If $2^{N-1} < |I| \leqslant 2^N$ $(N \in Z)$ then there is a $K \in N$ such that $I \subset [K2^N, (K+2) 2^N]$. Set

$$\mathsf{h}_i(t) = \tfrac{1}{4} \left(\mathsf{h}(t) - 2^{-N} \int_{(K+i) 2^N}^{(K+i+1) 2^N} \mathsf{h} \right) \chi_{[(K+i) 2^N, (K+i+1) 2^N)}(t),$$

where $i = 0, 1$. Then the h_i's are dyadic atoms. It follows from $\int_I \mathsf{h} = 0$ that $2^{-N} \int_{K2^N}^{(K+1) 2^N} \mathsf{h} = -2^{-N} \int_{(K+1) 2^N}^{(K+2) 2^N} \mathsf{h}$. Therefore, if

$$h_2(t) = \left(2^{-N} \int_{K2^N}^{(K+1) 2^N} \mathsf{h} \right) \left(\chi_{[K2^N, (K+1) 2^N)} - \chi_{[(K+1) 2^N, (K+2) 2^N)} \right)$$

then there exists $|\delta| \leqslant 4$ such that, using the notation of *Lemma 4.2*, $h_2 = \delta \mathsf{h}_{K, N}$. Thus h can be decomposed as $\mathsf{h} = \delta \mathsf{h}_{K, N} + 4\mathsf{h}_0 + 4\mathsf{h}_1$. ∎

For the proof of *Theorem 3.2* we need the following Lemma. Before stating it we redefine the *cosine* system by $\psi_t(x) = \sqrt{\tfrac{2}{\pi}} \cos tx$ $(t, x \in R^+)$ along with the corresponding Dirichlet kernel in order that we will be able to treat the *Walsh* and *cosine* cases uniformly.

LEMMA 4.3.   *Let* Ψ *denote the cosine or the* Walsh *system on* R^+. *Let* $\mathsf{h} \in \mathscr{A}_{R^+}$ *and define* $h \in L^1(R^+)$ *by* $h(x) = \int_{R^+} \mathsf{h}(t) D_t^\Psi(x) \, dt$ $(t \in R^+)$. *Then*

$$\hat{h}^\Psi(u) = \int_u^\infty \mathsf{h}(t) \, dt \qquad (u \in R^+).$$

Proof.   We note that $h \in L^1(R^+)$ follows from *Theorem 3.3*. By definition

$$\hat{h}^\Psi(u) = \int_{R^+} h(x) \, \psi_u(x) \, dx = \lim_{n \to \infty} \int_0^{2^n} h(x) \, \psi_u(x) \, dx \qquad (u \in R^+).$$

Using the definitions of h and D_t^Ψ $(t \in R^+)$ we can write

$$\int_0^{2^n} h(x) \, \psi_u(x) \, dx = \int_0^{2^n} \int_0^\infty \int_0^t \mathsf{h}(t) \, \psi_s(x) \, \psi_u(x) \, ds \, dt \, dx.$$

Recall that h vanishes outside of a compact interval. Consequently, we may apply Fubini's theorem to obtain

$$\int_0^{2^n} h(x)\,\psi_u(x)\,dx = \int_0^\infty \mathsf{h}(t) \int_0^t \int_0^{2^n} \psi_s(x)\,\psi_u(x)\,dx\,ds\,dt.$$

Let us evaluate $\int_0^t \int_0^{2^n} \psi_s(x)\,\psi_u(x)\,dx\,ds$. First we consider the Walsh case. It follows from the definition (11) that $w_x(y) = w_y(x)$ $(x, y \in \mathbf{R}^+)$. Then by (12) we have

$$\int_0^{2^n} w_s(x)\,w_u(x)\,dx = D_{2^n}^W(s \dot{+} u)$$

for a.e. $s \in \mathbf{R}^+$. Since $D_{2^n}^W = 2^n \chi_{[0,\,2^{-n})}$ $(n \in \mathbf{Z})$ (see [19]) we obtain

$$\int_0^{2^n} w_s(x)\,w_u(x)\,dx = 2^n \chi_{[k2^{-n},\,(k+1)\,2^{-n})}(s)$$

for a.e. $s \in \mathbf{R}^+$, where k is defined by $u \in [k2^{-n}, (k+1)\,2^{-n})$. Consequently,

$$\int_0^t \int_0^{2^n} w_s(x)\,w_u(x)\,dx\,ds = \begin{cases} 0 & t < k2^{-n}, \\ 2^n t - k & k2^{-n} \leqslant t < (k+1)\,2^{-n}, \\ 1 & t > (k+1)\,2^{-n}. \end{cases}$$

Let now Ψ be the *cosine* system. Then

$$\int_0^{2^n} \cos(sx)\cos(ux)\,dx = \frac{1}{\pi}\left(\frac{\sin 2^n(u+s)}{u+s} + \frac{\sin 2^n(u-s)}{u-s} \right).$$

By changing variables we obtain

$$\int_0^t \frac{\sin 2^n(u \pm s)}{u \pm s}\,ds = \pm \int_{2^n y}^{2^n(u \pm t)} \frac{\sin y}{y}\,dy.$$

Thus

$$\int_0^t \int_0^{2^n} \cos(sx)\cos(ux)\,dx\,ds = \frac{1}{\pi} \int_{2^n(u-t)}^{2^n(u+t)} \frac{\sin y}{y}\,dy.$$

We can conclude that

$$\lim_{n \to \infty} \int_0^t \int_0^{2^n} \psi_s(x)\,\psi_u(x)\,dx\,ds = \chi_{[u,\,\infty)}(t) \qquad (t \neq u)$$

uniformly for $t \in [0, u-\delta] \cup [u+\delta, \infty)$ $(\delta > 0)$, and that

$$\left| \int_0^t \int_0^{2^n} \psi_s(x) \, \psi_u(x) \, dx \, ds \right| \leqslant C \qquad (t > 0, n \in \mathbf{N})$$

for both the *cosine* and the Walsh systems. Therefore, we have

$$\left| \int_u^\infty \mathsf{h}(t) \, dt - \int_0^\infty \mathsf{h}(x) \, \psi_u(x) \, dx \right|$$

$$= \lim_{n \to \infty} \left| \int_0^\infty \mathsf{h}(t) \left(\chi_{[u, \infty)}(t) - \int_0^t \int_0^{2^n} \psi_s(x) \, \psi_u(x) \, dx \, ds \right) dt \right|$$

$$\leqslant \int_{[0, u-\delta] \cup [u+\delta, \infty)} |\mathsf{h}(t)| \lim_{n \to \infty} \left| \chi_{[u, \infty)}(t) - \int_0^t \int_0^{2^n} \psi_s(x) \, \psi_u(x) \, dx \, dx \right| dt$$

$$+ C \int_{u-\delta}^{u+\delta} |\mathsf{h}(t)| \, dt$$

$$\leqslant 2C\delta \, \|\mathsf{h}\|_{L^\infty(\mathbf{R}^+)} \qquad (\delta > 0).$$

Consequently, $\hat{\mathsf{h}}^\psi(u) = \int_u^\infty \mathsf{h}(t) \, dt$. ∎

Proof of Theorem 3.2. If Ψ denotes the *cosine* system then similarly to *Lemma 4.3* the ψ_t's are defined as $\psi_t(x) = \sqrt{\frac{2}{\pi}} \cos tx$ $(t, x \in \mathbf{R}^+)$. Set

$$g(x) = \lim_{u \to \infty} \int_0^u f(t) \, \psi_x(t) \, dt \qquad (x > 0). \tag{16}$$

First we show that g is well defined. By definition $\psi_t(x) = \psi_x(t)$, and $D_t^\psi(x) = \int_0^t \psi_s(x) \, ds = \int_0^t \psi_x(s) \, ds$ $(x, t \in \mathbf{R}^+)$. Then integration by parts yields

$$\int_0^u f(t) \, \psi_x(t) \, dt = [f(t) \, D_t^\psi(x)]_0^u - \int_0^u f'(t) \, D_t^\psi(x) \, dt.$$

It is known (see [19] and [4]) that $|D_t^\psi(x)| \leqslant C(1/x)$ $(t, x \in \mathbf{R}^+, x > 0)$. Since $D_0 \equiv 0$ it follows from $\lim_{t \to \infty} f(t) = 0$ that $\lim_{u \to \infty} [f(t) \, D_t^\psi(x)]_0^u = 0$. Moreover, $f' \in L^1(\mathbf{R}^+)$, and again $|D_t^\psi(x)| \leqslant C(1/x)$ imply that the function $f'(t) \, D_x^\psi(t)$ is integrable with respect to t for any $x > 0$. Consequently, g is well defined and

$$g(x) = -\int_0^\infty f'(t) \, D_t^\psi(x) \, dt \qquad (0 < x < \infty).$$

Since $f' \in \mathbf{H}_{R^+}$ we conclude by *Theorem 3.3* that g is integrable, and $\|g\|_{L^1(R^+)}$ $\leqslant C \|f'\|_{\mathbf{H}_{R^+}}$.

In order to prove $f = \hat{g}^{\psi}$ we will write g in another form. Namely, let $\sum_{k=0}^{\infty} \beta_k \mathsf{h}_k$ be an atomic decomposition of f'. Then we have by *Theorem 3.3* that the series

$$\sum_{k=0}^{\infty} \beta_k \int_0^{\infty} \mathsf{h}_k(t) \, D_t^{\psi} \, dt$$

converges in $L^1(R^+)$. Since $|D_t^{\psi}(x)| \leqslant C(1/x)$ $(t, x \in R^+, x > 0)$, $f' \in L^1(R^+)$, and $f' = \sum_{k=0}^{\infty} \beta_k \mathsf{h}_k$ a.e. we conclude by Lebesgue's theorem on integration that

$$g(x) = -\sum_{k=0}^{\infty} \beta_k \int_0^{\infty} \mathsf{h}_k(t) \, D_t^{\psi}(x) \, dt \qquad (x > 0).$$

Thus

$$\hat{g}^{\psi}(u) = -\sum_{k=0}^{\infty} \beta_k \int_0^{\infty} \int_0^{\infty} \mathsf{h}_k(t) \, D_t^{\psi}(x) \, \psi_u(x) \, dx \, dt \qquad (u \in R^+).$$

Let us apply *Lemma 4.3* to obtain

$$\hat{g}^{\psi}(u) = -\sum_{k=0}^{\infty} \beta_k \int_0^{\infty} \mathsf{h}_k(t) \, dt = -\int_u^{\infty} \sum_{k=0}^{\infty} \beta_k \mathsf{h}_k(t) \, dt$$

$$= -\int_u^{\infty} f'(t) \, dt = f(u) \qquad (u \in R^+). \quad \blacksquare$$

REFERENCES

1. B. Aubertin and J. Fournier, Integrability theorems for trigonometric series, *Studia Math.* **107** (1993), 33–59.
2. R. P. Boas, Absolute convergence and integrability of trigonometric series, *J. Rational Mech. Anal.* **5** (1956), 621–632.
3. M. Buntinas and N. Tanović-Miller, New integrability and L^1 convergence classes for even trigonometric series II, "Approximation Theory," Coll. Math. Soc. Janos Bolyai (J. Szabados and K. Tandori, Eds.), Vol. 68, pp. 103–125, North-Holland, Amsterdam, 1991.

4. P. L. Butzer and R. Nessel, "Fourier Analysis and Approximation," Birkhäuser, Basel/Stuttgart, 1971.
5. R. R. Coifman, A read variable characterization of H^p, *Studia Math.* **51** (1974), 269–274.
6. R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–645.
7. S. Fridli, An inverse Sidon type inequality, *Studia Math.* **105** (1993), 283–308.
8. S. Fridli, An inverse Sidon type inequality for the Walsh system, *J. Math. Anal. Appl.* **193** (1995), 715–736.
9. S. Fridli, Integrability and L^1-convergence of trigonometric and Walsh series, *Ann. Univ. Sci. Budapest. Sect. Comput.* **16** (1996), 149–172.
10. S. Fridli, On the integrability of Walsh–Fourier transform, *Math. Pannon.* **10** (1999), 93–102.
11. S. Fridli, Transition from the dyadic to the real nonperiodic Hardy space, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* **61** (2000), 1–8, electronic.
12. Dang Vu Giang and F. Móricz, On the L^1 theory of Fourier transforms and multipliers, *Acta Sci. Math. (Szeged)* **61** (1995), 293–304.
13. B. S. Kašin and A. A. Saakjan, "Orthogonal Series," Nauka, Moscow, 1984.
14. E. Liflyand, On asymptotics of Fourier transform for functions of certain classes, *Anal. Math.* **19** (1993), 151–168.
15. E. Liflyand, A Fourier transform approach to the integrability of trigonometric series, *in* "Theory of Approximation of Functions," Trudi IPMM NAN Ukraine, Vol. 3, pp. 134–145, 1998.
16. E. Liflyand, A family of function spaces and multipliers, *Israel Math. Conf. Proc.* **13** (1999), 141–149.
17. F. Shipp, Sidon-type inequalities, *in* "Approximation Theory" (G. A. Anastassiou, Ed.), Lecture Notes in Pure and Applied Mathematics, Vol. 138, pp. 421–436, Marcel Dekker, New York.
18. F. Schipp and L. Szili, Sidon-type inequalities for Legendre polynomials, *Acta Math. Hungar.* **68** (1995), 253–267.
19. F. Schipp, W. R. Wade, and P. Simon, "Walsh Series," Hilger, Bristol/New York, 1990.
20. S. A. Telyakovskiĭ, Integrability conditions of trigonometric series and their applications to the study of linear methods of summing Fourier series, *Izv. Akad. Nauk. SSSR, Ser. Mat.* **28** (1964), 1209–1236. [In Russian]